Published by Institute of Physics Publishing for SISSA/ISAS

Received: *November 5, 2004*
Accepted: *January 24, 2005*

Matrix-model dualities in the collective field formulation

Ivan Andrić and Danijel Jurman

Theoretical Physics Division, Rudjer Bošković Institute
P.O. Box 180, HR-10002 Zagreb, Croatia
E-mail: iandric@irb.hr, djurman@irb.hr

Abstract: We establish a strong-weak coupling duality between two types of free matrix models. In the large-N limit, the real-symmetric matrix model is dual to the quaternionic-real matrix model. Using the large-N conformal invariant collective field formulation, the duality is displayed in terms of the generators of the conformal group. The conformally invariant master hamiltonian is constructed and we conjecture that the master hamiltonian corresponds to the hermitian matrix model.

Keywords: Field Theories in Lower Dimensions, Conformal and W Symmetry, Matrix Models.

 http://jhep.sissa.it/archive/papers/jhep012005039/jhep012005039.pdf

Contents

1. Introduction

Single hermitian $N \times N$ matrix quantum mechanics is connected with $1+1$ dimensional strings via the double-scaling limit [1, 2, 3]. Recent progress in this field reinterpreted matrix quantum mechanics as a theory of N D0-branes [4, 5].

Two approaches in which matrix models are analysed provide two interpretations. In the first approach, the N eigenvalues of a matrix are treated as fermions moving classically in the inverted harmonic oscillator potential [6]. A single eigenvalue excitation over the filled Fermi sea has been interpreted as a D0-brane and the matrix degrees of freedom have been interpreted as an open tachyon field on the D0-branes. In the second approach, the bosonization of fermions is performed by introducing collective field which represents the density of eigenvalues [2, 7]. In this picture, the collective field describes the closed string degrees of freedom and small deformations of the Fermi sea are described by excitations of the massless scalar particle. An important phenomenon occurs: the space of eigenvalues provides a new space dimension in which the string moves, and a holographic description arises [8, 9]. Since closed strings are part of gravity theory, the duality between the $0+1$ non-gravitational theory (open strings) and the $1+1$ theory which contains gravity (closed strings) is holographic. Owing to the correspondence we expect that calculations in string theory can be performed in the simpler matrix theory. Such a programme has already been used to study cosmology [10] and particle production in cosmology [11]. There are attempts to treat black holes in a similar way [12].

Correspondence between various matrix models and string theories goes beyond the hermitian matrix model. It has been shown that unoriented string theories correspond to real-symmetric and quaternionic-real matrix models, depending on the type of the orientifold projection under study [13]. Analysing themodynamical propeties of these two models in the aforementioned first approach, the duality has been established. The same type of duality has emerged in the study of the Calogero-Moser models [14] to which matrix models reduce in the first approach.

JHEP01(2005)039

In this paper we show the appearance of duality between the real-symmetric and the quaternionic-real matrix model in the second approach. The first model is invariant to the SO(N) group and the second to the Sp(N) group of transformations. The paper is organised as follows. In section 2 we develop a general formalism to express a particular matrix model in the collective field formalism. In section 3 we discuss the invariance of the collective field Lagrangian descending from the matrix model. Duality relations are displayed in terms of the generators of the conformal symmetry group and the master hamiltonian for dual systems is constructed preserving conformal invariance. In section 4, we consider a connection of the master hamiltonian with the hamiltonian of the hermitian matrix model. In conclusion we summarize the main results.

2. Matrix model and the collective-field hamiltonian

The dynamics of the one-matrix model is defined by the action reference [15, 1]

$$S = \int dt \left(\frac{1}{4} \operatorname{Tr} \dot{M}^2(t) - V(M) \right) , \tag{2.1}$$

with the matrix M of the form

$$M = \sum_{\alpha=0}^{3} m_\alpha \otimes e^{(\alpha)} , \qquad \alpha = 0, 1, 2, 3 , \tag{2.2}$$

where real, $N \times N$ matrices m_α's have the following properties:

$$m_0^T = m_0 , \qquad m_l^T = -m_l , \qquad l = 1, 2, 3 , \tag{2.3}$$

and elementary quaternions $e^{(\alpha)}$'s are represented by 2×2 matrices [16]

$$e^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \qquad e^{(1)} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} , \qquad e^{(2)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} , \qquad e^{(3)} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} . \tag{2.4}$$

From the definition (2.4) we obtain commutation relations

$$\left[e^{(i)}, e^{(j)} \right] = 2\epsilon_{ijk} e^{(k)} , \tag{2.5}$$

where ϵ_{ijk} is the totally antisymmetric with respect to the permutations of the indices and $\epsilon_{123} = 1$.

We consider three types of matrices: real-symmetric, hermitian and quaternionic-real. The factor 1/4 in the action (2.1) has been introduced to make possible a unique treatment of all three models. With the definitions given above, equation (2.2) represents a quaternionic-real matrix. Taking $m_i = 0$ reduces (2.1) to the familiar expression

$$S = \frac{1}{2} \int dt \operatorname{Tr} \dot{R}^2 , \tag{2.6}$$

where $R = m_0$ is the real-symmetric matrix and if we take $m_1 = m_2 = 0$, the expression (2.1) reduces to

$$S = \frac{1}{2} \int dt \operatorname{Tr} \dot{G}^2 , \tag{2.7}$$

where $G = m_0 + i m_3$ is the hermitian matrix.

JHEP01(2005)039

To analyse a matrix model in the large-N limit, we introduce the collective field variables

$$\phi_k(t) = \frac{1}{2}\,\mathrm{Tr}\, e^{-ikM(t)}\,. \tag{2.8}$$

This otherwise over-complete set of variables becomes complete in the large-N limit and we can express the action (2.1) in terms of $\phi_k(t)$'s. The general procedure is developed in [7] and for the quaternionic-real model it was done by expansion and resummation in k [17]. Here we present a similar method based on the tensor product of algebras, which can be generalized for other applications.

Expressed in terms of collective field in coordinate space

$$\phi(x,t) = \frac{1}{2}\int \frac{dk}{2\pi} e^{ikx}\phi_k(t)\,, \tag{2.9}$$

the free matrix hamiltonian is

$$H = \frac{1}{2}\int\int dx dy \Omega[\phi; x, y]\pi(x)\pi(y) - \frac{i}{2}\int dx\omega[\phi; x]\pi(x)\,. \tag{2.10}$$

Whereas $\pi(x)$ in (2.10) is the canonical conjugate of $\phi(x)$, $\Omega[\phi; x, y]$ and $\omega[\phi; x]$ are to be determined by transformation from quantum mechanics to collective field theory. The factor 1/2 in the definitions (2.8) and (2.9) is present because of the normalization condition

$$\int dx\phi(x) = N\,, \tag{2.11}$$

where N is the number of independent eigenvalues of the matrix M. Again, the expressions (2.8) and (2.9) reduce to familiar expressions without the factor 1/2 in the cases of symmetric and hermitian matrices.

In order to formulate collective field theory for a matrix model, we have to calculate $\Omega[\phi; x, y]$ and $\omega[\phi; x]$. For this purpose, we establish some preliminary identities. First we notice that if M is a quaternionic-real (real-symmetric, hermitian) matrix, then M^n is also a quaternionic-real (real-symmetric, hermitian) matrix

$$M^n \equiv \sum_{\alpha=0}^{3} m_\alpha(n)\otimes e^{(\alpha)}\,, \qquad m_0^T(n) = m_0(n)\,, \qquad m_l^T(n) = -m_l(n)\,, \quad l = 1, 2, 3\,. \tag{2.12}$$

This statement is easily proved by induction, collecting appropriate terms in

$$M^{n+1} = \frac{1}{2}(M^n M + M M^n)\,. \tag{2.13}$$

Defining the decomposition of the matrix $\exp(-isM)$ in terms of quaternions

$$[s] \equiv e^{-isM} \equiv \sum_{\alpha=0}^{3}[s]_\alpha \otimes e^{(\alpha)} \tag{2.14}$$

and using (2.12) we conclude that matrices $[s]_\alpha$'s have the following properties:

$$[s]_0^T = [s]_0\,, \qquad [s]_l^T = -[s]_l\,, \quad l = 1, 2, 3\,. \tag{2.15}$$

Now we introduce further decomposition of the symmetric and antisymmetric matrices in (2.3):

$$
\begin{aligned}
m_0 &= \sum_{i,j=1,i\leq j}^{N} m_0^{ij} h_{ij}^{+}\,, \\
m_l &= \sum_{i,j=1,i<j}^{N} m_l^{ij} h_{ij}^{-}\,, \qquad l=1,2,3\,,
\end{aligned} \tag{2.16}
$$

where $h_{ij}^{\pm}$ are elementary matrices with elements at the m-th row and the n-th column defined by

$$
\left[h_{ij}^{\pm}\right]_{mn} = \delta_{im}\delta_{jn} \pm \delta_{in}\delta_{jm}\,. \tag{2.17}
$$

From the definitions of $h_{ij}^{\pm}$ we obtain the following trace rules:

$$
\begin{aligned}
\sum_{i,j=1}^{N} \mathrm{Tr}(X h_{ij}^{\pm} X' h_{ij}^{\pm}) &= 2[\mathrm{Tr}(X X'^{T}) \pm (\mathrm{Tr}\, X)(\mathrm{Tr}\, X')]\,, \\
\sum_{i,j=1}^{N} \mathrm{Tr}(X h_{ij}^{\pm})\, \mathrm{Tr}(X' h_{ij}^{\pm}) &= 2\, \mathrm{Tr}(X X'^{T} \pm X X')\,.
\end{aligned} \tag{2.18}
$$

After these preliminary remarks we are in a position to present the calculation of the relevant functionals $\Omega[\phi;x,y]$ and $\omega[\phi;x]$. Passing from the lagrangian formulation to the hamiltonian, after performing quantization, the transformation to the collective field hamiltonian is obtained by application of the chain rule

$$
\frac{\partial}{\partial m_\alpha^{ij}} \to \int dx \frac{\partial\phi(x)}{\partial m_\alpha^{ij}} \frac{\delta}{\delta\phi(x)}\,. \tag{2.19}
$$

Using (2.19), for $\Omega[\phi;x,y]$ and $\omega[\phi;x]$ in (2.10) we find

$$
\Omega[\phi;x,y] = \frac{1}{2}\left[\sum_{i,j=1,i\leq j}^{N}(1+\delta_{ij})\frac{\partial\phi(x)}{\partial m_0^{ij}}\frac{\partial\phi(y)}{\partial m_0^{ij}} + \sum_{l=1}^{3}\sum_{i,j=1,i<j}^{N}\frac{\partial\phi(x)}{\partial m_l^{ij}}\frac{\partial\phi(y)}{\partial m_l^{ij}}\right], \tag{2.20}
$$

$$
\omega[\phi;x] = -\frac{1}{2}\left(\sum_{i,j=1,i\leq j}^{N}(1+\delta_{ij})\frac{\partial^{\,2}\phi(x)}{\partial m_0^{ij\,2}} + \sum_{l=1}^{3}\sum_{i,j=1,i<j}^{N}\frac{\partial^{\,2}\phi(x)}{\partial m_l^{ij\,2}}\right). \tag{2.21}
$$

Performing the first summation in (2.20) we obtain

$$
\begin{aligned}
\frac{1}{2}\sum_{i,j=1,i\leq j}^{N}(1+\delta_{ij})\frac{\partial\phi(x)}{\partial m_0^{ij}}\frac{\partial\phi(y)}{\partial m_0^{ij}} &= \frac{1}{8}\int\!\!\int \frac{dkdk'}{(2\pi)^2}(\partial_x e^{ikx})(\partial_y e^{ik'y}) \times \\
&\quad \times \sum_{i,j=1,i\leq j}^{N}(1+\delta_{ij})\,\mathrm{Tr}\left\{[k]\left(1-\frac{1}{2}\delta_{ij}\right)(h_{ij}^{+}\otimes e_0)\right\} \times \\
&\quad \times \mathrm{Tr}\left\{[k']\left(1-\frac{1}{2}\delta_{ij}\right)(h_{ij}^{+}\otimes e_0)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\int\int \frac{dkdk'}{(2\pi)^2}(\partial_x e^{ikx})(\partial_y e^{ik'y}) \times \\
&\times \sum_{i,j=1}^{N} \mathrm{Tr}\left\{[k]_0 h^+_{ij}\right\} \mathrm{Tr}\left\{[k']_0 h^+_{ij}\right\} \\
&= \int\int \frac{dkdk'}{(2\pi)^2}(\partial_x e^{ikx})(\partial_y e^{ik'y})\, \mathrm{Tr}\left\{[k]_0 [k']_0\right\}
\end{aligned} \tag{2.22}
$$

and analogously for the other sums in (2.20)

$$
\frac{1}{2}\sum_{i,j=1,i<j}^{N} \frac{\partial\phi(x)}{\partial m_l^{ij}}\frac{\partial\phi(y)}{\partial m_l^{ij}} = -\int\int \frac{dkdk'}{(2\pi)^2}\left(\partial_x e^{ikx}\right)\left(\partial_y e^{ik'y}\right) \mathrm{Tr}\left\{[k]_l\,[k']_l\right\}. \tag{2.23}
$$

The first step in (2.22) is obtained from the definitions (2.8) and (2.9), and we have only rewritten the multiplications by k and k' as appropriate derivatives. The second step is obtained by use of the trace property for the tensor product of matrices

$$
\mathrm{Tr}[(A\otimes a)(B\otimes b)] = \mathrm{Tr}(AB)\,\mathrm{Tr}(ab) \tag{2.24}
$$

and by the orthogonality of $e^{(\alpha)}$'s:

$$
\mathrm{Tr}\, e^{(\alpha)}e^{(\beta)} = 2\eta^{\alpha\beta}\,, \eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.25}
$$

The third, last step in (2.22) is obtained by use of (2.18), (2.12) and by rewriting the sum over $i \leq j$ as the sum over $i \neq j$. Collecting the partial results (2.22) and (2.23) we obtain for $\Omega[\phi;x,y]$

$$
\Omega[\phi;x,y] = \frac{1}{2}\partial^2_{xy}\int \frac{dkdk'}{(2\pi)^2} e^{ikx}e^{ik'y}\, \mathrm{Tr}\, e^{-i(k+k')M}. \tag{2.26}
$$

The calculation of $\omega[\phi;x]$ is performed in a similar way once the second derivatives have been rewritten in a suitable form, using the identity

$$
\partial^2\, \mathrm{Tr}\, e^{-ikM} = -k^2\int_0^1 d\beta\, \mathrm{Tr}\left[e^{-ik\beta M}(\partial M)e^{-ik(1-\beta)M}(\partial M)\right]. \tag{2.27}
$$

As an example, for the first sum in (2.21) we have

$$
\begin{aligned}
-\frac{1}{2}\sum_{i,j=1,i\leq j}^{N}(1+\delta_{ij})\frac{\partial^2\phi(x)}{\partial m_0^{ij\,2}} &= \frac{1}{4}\int\int_0^1 \frac{dkd\beta}{2\pi}k^2 e^{ikx} \times \\
&\times \sum_{i,j=1,i\leq j}^{N}(1+\delta_{ij})\, \mathrm{Tr}\left\{e^{-ik\beta M}\frac{\partial M}{\partial m_0^{ij}}e^{-ik(1-\beta)M}\frac{\partial M}{\partial m_0^{ij}}\right\}.
\end{aligned} \tag{2.28}
$$

From this point the calculation is lenghtly but straigthforward and the final result is

$$
\omega[\phi;x] = \frac{1}{2}\int\int_0^1 \frac{dkd\beta}{2\pi}k^2 e^{ikx}\left(-\,\mathrm{Tr}\, e^{-ikM} + \mathrm{Tr}\, e^{-i\beta kM}\, \mathrm{Tr}\, e^{-i(1-\beta)kM}\right). \tag{2.29}
$$

Substituing the inverse of (2.9)

$$\mathrm{Tr}\, e^{-ikM} = 2\int dx e^{-ikx}\phi(x) \tag{2.30}$$

into (2.26) and (2.29), we find

$$\begin{aligned}\Omega[\phi;x,y] &= \partial^2_{xy}\left[\delta(x-y)\phi(y)\right] \\ \omega[\phi;x] &= (\lambda-1)\partial_x^2\phi(x) + 2\lambda\partial_x\phi(x)\fint dy\frac{\phi(y)}{x-y}\,.\end{aligned} \tag{2.31}$$

The parameter λ in (2.31) determines the number of independent matrix elements n_λ in the case of real-symmetric, hermitian and quaternionic-real matrices:

$$n_\lambda = \lambda N(N-1) + N \tag{2.32}$$

and $\lambda = 1/2$, 1, 2, respectively. λ is called the statistical parameter because it enters in the exponent of the integration measure over matrices and therefore in the exponent of the prefactor in the wave function [16]. If we exchange two eigenvalues, the wave function changes its phase by $e^{i\pi\lambda}$. For $\lambda = 1$, the statistics of the matrix eigenvalues are fermionic, for $\lambda = 0$ (diagonal matrix) bosonic and for $\lambda = 1/2$ and $\lambda = 2$ we have an exclusion type of statistics [18].

Finally, after hermitization [7] of (2.10) we obtain the collective field hamiltonian

$$\begin{aligned}H =& \frac{1}{2}\int dx\phi(x)(\partial_x\pi)^2 + \frac{1}{2}\int dx\phi(x)\left(\frac{\lambda-1}{2}\frac{\partial_x\phi(x)}{\phi} + \lambda\fint dy\frac{\phi(y)}{x-y}\right)^2 - \\ &-\mu\int dx\phi(x) + \int dx\phi(x)V(x) - \frac{\lambda-1}{4}\int dx\partial_x^2\delta(x-y)|_{y=x} - \\ &-\frac{\lambda}{2}\fint dx\partial_x\frac{1}{x-y}\bigg|_{y=x}\,,\end{aligned} \tag{2.33}$$

where the term with the Lagrange multipliler μ has been added because of the constraint (2.11). The last two terms in (2.33), which are singular, do not contribute in the leading order in N [19].

3. Conformal invariance and duality

In this section we find generators of symmetries of the action defined by the lagrangian density corresponding to the hamiltonian (2.33)

$$\mathcal{L}(\phi,\dot\phi) = \frac{1}{2}\frac{(\partial_x^{-1}\dot\phi)^2}{\phi} - \frac{1}{2}\phi\left[\frac{(\lambda-1)}{2}\frac{\partial_x\phi}{\phi} + \lambda\fint dy\frac{\phi(y)}{x-y}\right]^2, \tag{3.1}$$

where ∂_x^{-1} is short for $\int^x dy\dot\phi(y)$.

However, let us first establish a fundamental property of the collective field lagrangian descending from the matrix models. In order to formulate string theory, we need to analyse the matrix model in the critical potential. In the cubic theory (the hermitian matrix model), it has been shown that the kinetic term induces the harmonic potential [20, 21]

$$\int dxdt \frac{\left(\partial_x^{-1}\dot{\phi}\right)^2}{\phi} = \int dx'dt' \left[\frac{\left(\partial_{x'}^{-1}\dot{\phi}'\right)^2}{\phi'} + x'^2\phi'(x',t')\right] \tag{3.2}$$

through a coordinate reparametrization and field rescaling

$$x = \frac{x'}{\cosh t'}\,, \qquad t = \tanh t'\,, \qquad \phi(x,t) = \phi(x',t')\cosh t'\,. \tag{3.3}$$

Similarly, it can be shown that the second term in the lagrangian (3.1) remains invariant and therefore all three matrix models have background independence. This property enables us to concentrate the discussion on the free models.

To display the duality of the matrix models, we need infinitesimal generators of the symmetry of the action defined by the collective field lagrangian (3.1). The symmetry transformations are the global conformal reparametrisations of time which leave the action invariant, but are not the symmetries of the lagrangian. Therefore, a Noether theorem is needed with an additional term owing to the change of the form of the lagrangian:

$$\delta S = S' - S = \int dx'dt'\mathcal{L}\left[\phi'(x',t'),\dot{\phi}'(x',t')\right] - \int dxdt\mathcal{L}\left[\phi(x,t),\dot{\phi}(x,t)\right] = \int dt\frac{dA}{dt}\,, \tag{3.4}$$

where A is a functional of ϕ and $\dot{\phi}$ to be determined. On the other hand, the change of the action owing to the infinitesimal symmetry transformation $\delta\phi$, obtained by use of the Euler-Lagrange equation of motion, is

$$\delta S = \int\int dtdx\left(\frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\dot{\phi}}\delta\dot{\phi}\right) = \int dt\frac{d}{dt}\left(\int dx\frac{\delta\mathcal{L}}{\delta\dot{\phi}}\delta\phi\right). \tag{3.5}$$

This change of action should be equal to (3.4) and for the conserved quantity we obtain

$$Q = \int dx\frac{\delta\mathcal{L}}{\delta\dot{\phi}}\delta\phi - A\,. \tag{3.6}$$

We show that the action determined by the lagrangian density (3.1) possesses three kinds of symmetry: time translation, scaling and special conformal tranformation. The infinitesimal forms of these transformations are

$$t' = t - \epsilon t^n\,, \tag{3.7}$$

for $n = 0, 1, 2$, respectively. Under these transformations, the space coordinate x, the field $\phi(x,t)$ and the space-time volume element $dtdx$ transform according to

$$\begin{aligned} x' &= \left(\frac{\partial t'}{\partial t}\right)^{d_x} x\,,\\ \phi'(x',t') &= \left(\frac{\partial t'}{\partial t}\right)^{d_\phi}\phi(x,t)\,,\\ dx'dt' &= \left(\frac{\partial t'}{\partial t}\right)^{d_x+1} dxdt\,. \end{aligned} \tag{3.8}$$

Dimensions are determined to be $d_x = 1/2$ and $d_\phi = -1/2$. Performing the infinitesimal transformation (3.7), from (3.8) we obtain

$$\delta\phi(x,t) = \phi'(x,t) - \phi(x,t) = (-d_\phi n t^{n-1} + d_x n t^{n-1} x\partial_x + t^n \partial_t)\phi(x,t)\,. \tag{3.9}$$

Introducing

$$\partial_x \pi = \partial_x \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = -\frac{1}{\phi}\partial_x^{-1}\dot{\phi}\,, \tag{3.10}$$

we find for the first part of the conserved quantity (3.6)

$$\int dx \frac{\delta\mathcal{L}}{\delta\dot{\phi}}\delta\phi = \int dx \pi(x,t)\delta\phi = -\frac{n}{2}t^{n-1}\int dx x\phi(x)\partial_x\pi + t^n \int dx \frac{\left(\partial_x^{-1}\dot{\phi}\right)^2}{\phi} \tag{3.11}$$

and after some calculation, from the difference of the lagrangians (3.4) we obtain

$$A = -\frac{n(n-1)}{4}\int dx x^2\phi + \frac{t^n}{2}\int dx\mathcal{L}\,. \tag{3.12}$$

Substituing (3.11) and (3.12) in (3.6) we obtain for $n = 0, 1, 2$

$$\begin{aligned}
Q_0 &= H \equiv Q_T\,,\\
Q_1 &= -\frac{1}{2}\int dx\phi(x)\partial_x\pi(x) + tH \equiv Q_S\,,\\
Q_2 &= \frac{1}{2}\int dx x^2\phi(x) - t\int dx x\phi(x)\partial_x\pi(x) + \frac{t^2}{2}H \equiv Q_C\,.
\end{aligned} \tag{3.13}$$

These conserved quantities close the algebra of the conformal group in one dimension with respect to the classical Poisson brackets

$$\{Q_T, Q_S\}_{PB} = Q_T\,, \{Q_C, Q_S\}_{PB} = -Q_C\,, \{Q_T, Q_C\}_{PB} = 2Q_T\,. \tag{3.14}$$

Performing the quantisation, simplifying by taking $t = 0$ and performing a similarity transformation, we obtain the generators used in reference [23]

$$\begin{aligned}
T_+[\phi,\lambda] &= -J^{-1/2}Q_T J^{1/2} = -\frac{1}{2}\int dx\phi(x)(\partial_x\pi(x))^2 - \frac{i}{2}\int dx\omega[\phi;x]\pi(x)\,,\\
T_- &= Q_C\,,\\
T_0 &= iJ^{-1/2}Q_S J^{1/2} = -\frac{1}{2}\left(i\int dx x\phi(x)\partial_x\pi(x) + E_0\right),
\end{aligned} \tag{3.15}$$

where the Jacobian J is determined by $\omega[\phi;x]$

$$\omega[\phi;x] = \partial_x\left(\phi(x)\frac{\delta \ln J}{\delta\phi(x)}\right). \tag{3.16}$$

The constant $E_0 = n_\lambda/2$ in (3.15) , where n_λ is given by (2.32), is the ground-state energy of the hamiltonian (2.33) with the additional harmonic interaction $V(x) = x^2/2$ known to be equivalent to the operator T_0 up to the similarity transformation [22]. We can interpret E_0 as the ground-state energy of the n_λ independent harmonic oscillators.

After establishing the representation of the su $(1,1)$ algebra

$$[T_+, T_-] = -2T_0 \,, \qquad [T_0, T_\pm] = \pm 2T_\pm \tag{3.17}$$

we summarise some known results. It has been shown in reference [23] that the eigenfunctionals of the hamiltonian (2.33) can be determined if the zero-energy eigenfunctionals are known

$$T_+[\phi;\lambda]P_m[\phi] = 0 \,, \qquad T_0[\phi;\lambda]P_m[\phi] = \mu_m P_m[\phi] \,. \tag{3.18}$$

The functional $J^{1/2}P_m$ is then the zero-energy eigenfunctional in accordance with the conformal invariance of the lagrangian. Owing to the spectrum generating algebra (3.17) the eigenfunctional of the energy E is the coherent state of the Barut-Girardello-type [24]. In order to show this, we define the operator

$$\hat{T} = -T_- \frac{1}{T_0 + \mu_c} \,, \tag{3.19}$$

which has the canonical commutation relation with T_+

$$\left[T_+, \hat{T}\right] = 1 \,, \tag{3.20}$$

and μ_c is determined by the eigenvalue of the Casimir operator $\hat{C}$

$$\mu_c = -\frac{1}{2} + \sqrt{\frac{1}{4} - \hat{C}} \,, \tag{3.21}$$

$$\hat{C} = T_- T_+ + T_0(T_0 + 1) \,. \tag{3.22}$$

Then, the coherent state $e^{E\hat{T}}P_m[\phi]$ is the eigenfunctional of T_+, and $J^{1/2}e^{E\hat{T}}P_m[\phi]$ of the hamiltonian (2.33). By applying $e^{E\hat{T}}$ to $P_m[\phi]$ and using (3.17), we obtain another form for continuum states with the eigenvalue E

$$e^{E\hat{T}}P_m[\phi] \sim T_-^{-(m+E_0-3/2)/2} Z_{m+E_0-3/2}\left(2\sqrt{ET_-}\right) P_m[\phi] \,, \tag{3.23}$$

where $Z_\alpha(x)$ stands for the Bessel function.

In addition to (3.23) there exist other solutions [21, 25, 26] in the case $\lambda \neq 1$ in the non-perturbative sector of the theory. They are soliton solutions and appear in the BPS limit, because the kinetic term in the hamiltonian is of order $1/N$ with respect to the positive definite second term in (2.33). The BPS limit leads to the first-order integro-differential equation and the solution describes the static tachyonic background with the energy proportional to the charge of the soliton. The non-BPS solutions are obtained from the Heisenberg equations of motions. These are moving soliton solutions [26, 21, 25] and for $\lambda = 1/2$, they describe holes in the background (condensate) and for $\lambda = 2$, lumps above the background, as has already been observed in the Calogero-Sutherland model [27].

The solitons of the collective field description are dual quasi-particles. To display this duality, we introduce a new field $m(x,t)$ describing quasi-particles, and this new field will enter in the prefactor of the wave functional. For the prefactor we take the continuum analogue of the prefactor used in the discrete case [14]:

$$V^\kappa[\phi, m] = e^{\kappa \int\int dx dy \phi(x) \ln|x-y| m(y)} \,. \tag{3.24}$$

JHEP01(2005)039

The duality is displayed by the following relations:

$$T_+[\phi,\lambda]V^\kappa[\phi,m] = \Bigg[-\frac{\lambda}{\kappa}T_+\left[m,\frac{\kappa^2}{\lambda}\right] + \frac{\kappa\pi^2}{2}\int dx\phi(x)m(x)(\lambda\phi(x)+\kappa m(x)) +$$
$$+\frac{(\lambda+\kappa)(\kappa-1)}{4}\iint dxdz\frac{m(z)\partial_x\phi(x)-\phi(x)\partial_z m(z)}{x-z}\Bigg]V^\kappa[\phi,m]\,, \quad (3.25)$$

$$T_0[\phi,\lambda]V^\kappa[\phi,m] = -\left(T_0\left[m,\frac{\kappa^2}{\lambda}\right] + \frac{E_0(N,\lambda)+E_0(M,\kappa^2/\lambda)+\kappa NM}{2}\right)V^\kappa[\phi,m]\,. \quad (3.26)$$

Here we have the manifest strong/weak coupling duality. If we interchange the fields $\phi(x)$ and $m(z)$, the coupling constant λ goes to κ^2/λ. The duality relations (3.25) and (3.26) are crucial. They enable us to construct new su $(1,1)$ generators for the system of particles and dual quasi-particles

$$\begin{aligned}
\mathcal{T}_+ &= T_+[\phi,\lambda] + \frac{\lambda}{\kappa}T_+\left[m,\frac{\kappa}{\lambda}\right] - \mathcal{H}_{int}\,,\\
\mathcal{H}_{int} &= \frac{(\lambda+\kappa)(\kappa-1)}{4}\int\int dxdz\frac{m(z)\partial_x\phi(x)-\phi(x)\partial_z m(z)}{x-z} +\\
&\quad +\frac{\kappa\pi^2}{2}\int dx\phi(x)m(x)(\lambda\phi(x)+\kappa m(x))\,,\\
\mathcal{T}_0 &= T_0[\phi,\lambda] + T_0\left[m,\frac{\kappa^2}{\lambda}\right],\\
\mathcal{T}_- &= T_-[\phi,\lambda] + \frac{\kappa}{\lambda}T_-\left[m,\frac{\kappa^2}{\lambda}\right].
\end{aligned} \quad (3.27)$$

We interpret the operator $\mathcal{T}_+$ in (3.27) as a non-hermitian hamiltonian for the system of particles and quasi-particles. After hermitisation of $\mathcal{T}_+$ we obtain the hermitian form

$$\mathcal{H}^M = H[\phi,\lambda] + \frac{\lambda}{\kappa}H\left[m,\frac{\kappa^2}{\lambda}\right] + \mathcal{H}_{int}\,, \quad (3.28)$$

where $H[\phi,\lambda]$ is the hamiltonian (2.33) and $H[m,\kappa^2/\lambda]$ is obtained from (2.33) by substituting m for ϕ and κ^2/λ for λ.

4. Master hamiltonian and the hermitian matrix model

In this section we argue that the master hamiltonian (3.28) corresponds to the hermitian matrix model. The starting point is the Lagrangian

$$L = \frac{1}{2}\,\mathrm{Tr}\,\dot{G}^2\,, \quad (4.1)$$

where G is a hermitian matrix. Following the usual procedure of quantization, from the lagrangian (4.1) we obtain the hamiltonian in coordinate representation

$$H_G = -\frac{1}{2}\partial_G^2 \equiv -\frac{1}{4}\sum_{i,j=1,i\leq j}^{N}(1+\delta_{ij})\frac{\partial}{\partial g^{ij}}\frac{\partial}{\partial g^{ij}}\,, \quad (4.2)$$

where g^{ij}'s are elements of the matrix G. Expressed in terms of the collective field, the hamiltonian (4.2) corresponds to the collective hamiltonian (2.33) with $\lambda = 1$.

JHEP01(2005)039

Next, we decompose the hermitian matrix into the sum of the symmetric and antisymmetric matrix, $G = R + iA$. This decomposition is followed by the decomposition of the hamiltonian (4.2)

$$H_G = -\frac{1}{2}\partial_G^2 = -\frac{1}{2}\partial_R^2 + \frac{1}{2}\partial_A^2 \,. \tag{4.3}$$

Now, if we restrict the wavefunction of the system to be dependent only on the eigenvalues of the symmetric matrix, by choosing collective field variables

$$\phi_k = \operatorname{Tr} e^{-ikR} \,, \tag{4.4}$$

we obtain the collective hamiltonian (2.33) with the parameter $\lambda = 1/2$. Using duality relation (3.25) for $\kappa = 1$, we construct the master hamiltonian (3.28) which contains the following: the hamiltonian (2.33) with parameter $\lambda = 1/2$ describing symmetric-matrix degrees of freedom, the hamiltonian (2.33) with parameter $\lambda = 2$ describing quaternionic-matrix degrees of freedom and the hamiltonian of the interaction between the field $\phi(x)$ and dual field $m(x)$.

As the next step, we argue that the hamiltonian (2.33) with $\lambda = 2$ describes a system with antisymmetric-matrix degrees of freedom. To show this, we have to find appropriate collective field variables for the antisymmetric matrix model. By naive generalisation, we might use

$$m_k = \operatorname{Tr} e^{-ikA} \,, \tag{4.5}$$

but this choice is misleading. As a first objection, we recall that the antisymmetric matrix has pure imaginary eigenvalues implying exponential behaviour of the variables (4.5). As a second objection, we notice that A^n is not antisymmetric matrix and this property implies that the set of elementary matrices from the decomposition of A is not complete. To resolve this puzzle, we use isomorphism between antisymmetric and quaternionic-real matrices. We write the antisymmetric matrix $(2n \times 2n)$ in terms of quaternions:

$$A = A_2 \otimes e^{(0)} - iA_3 \otimes e^{(1)} - R \otimes e^{(2)} + iA_1 \otimes e^{(3)} \,, \tag{4.6}$$

where

$$R^T = R \,, \qquad A_l^T = -A_l \tag{4.7}$$

and define the quaternionic-real matrix

$$Q = \frac{1}{2}\Big(\{e^{(2)}, A\} - i[e^{(2)}, A]\Big) = R \otimes e^{(0)} + A_1 \otimes e^{(1)} + A_2 \otimes e^{(2)} + A_3 \otimes e^{(3)} \,, \tag{4.8}$$

which has real eigenvalues and for which we know that Q^n is also the quaternionic-real matrix. Noticing that

$$-\operatorname{Tr} \dot{A}^2 = \operatorname{Tr} \dot{Q}^2 \to -\partial_A^2 = \partial_Q^2 \,, \tag{4.9}$$

then choosing the collective field variables

$$m_k = \frac{1}{2}\operatorname{Tr} e^{-ikQ} \tag{4.10}$$

and expressing ∂_Q^2 in terms of these we obviously end up with the hamiltonian (2.33) for $\lambda = 2$.

JHEP01(2005)039

It follows from the above discussion that starting from the hamiltonian (4.1) and restricting the wave function of the system to be dependent only on the eigenvalues of the symmetric matrix and the quaternionic matrix defined by (4.8), the corresponding collective field hamiltonian is

$$H = H_{1/2} + \frac{1}{2} H_2 \,, \tag{4.11}$$

where $H_{1/2}$ and H_2 are the collective field hamiltonians of the symmetric and quaternionic matrices, respectively. Notice that the factor 1/2 in the lagrangian (4.1) instead of 1/4 as in (2.1) introduces the factor 1/2 in front of H_2. Comparing (4.11) with (3.28), we see that (4.11) is equal to the master hamiltonian without interaction terms.

To show that the master hamiltonian indeed describes the hermitian matrix model we have to generate interaction term. This is achieved by extracting the prefactor $\Pi_{i,\alpha}(x_i - z_\alpha)$ (x_i and z_α are the eigenvalues of the symmetric and quaternionic matrices, respectively) from the wavefunction and defining new hamiltonian

$$\tilde{H}_G = \Pi_{i,\alpha}(x_i - z_\alpha)^{-1} H_G \Pi_{i,\alpha}(x_i - z_\alpha) \,. \tag{4.12}$$

This prefactor has been found in [14] to appear in the ground-state and it would be interesting to see whether it is connected with the integration measure. Physically, this prefactor prevents finding the particle and quasi-particle at the same point. Expressing this new hamiltonian in terms of the collective-fields (4.4) and (4.10), we obtain the master hamiltonian (3.28).

Finally, we see that duality relations enabled us to construct the master hamiltonian and to recover the degrees of freedom of the hermitian matrix, which were lost by choosing (4.4) as the collective field variables.

5. Conclusion

We have seen that su $(1,1)$ algebra generates the dynamical symmetry of the matrix models in the collective field approach. This algebra makes possible the construction of eigenfunctionals, an explicit display of duality relations between matrix models and the construction of the master hamiltonian in a conformally invariant way. We conjecture that the master hamiltonian with $\lambda = 1/2$ describes the hermitian matrix model. This gives deeper insight into the properties of the hermitian matrix model. In reference [21], the hermitian matrix model was analysed by the exact construction of eigenstates represented by the Young-tableaux. For the Yang-tableaux only with one column and only with one row, effective lagrangians were constructed, which correspond to the $\lambda = 1/2$ and $\lambda = 2$ hamiltonians (2.33) in our language. Further analysis of the dynamics of the master hamiltonian could give us more information on closed string states described in the collective field approach.

Acknowledgments

We would like to thank A. Jevicki for discussion and hospitality at Brown University, where part of this article has been done. We are also thankfull to L. Jonke for discussion and A. Kapustin for usefull insight. This work was supported by the Ministry of Science and Technology of the Republic of Croatia under contract No. 0098003.

References

[1] Several reviews including references to the original literature are
I.R. Klebanov, *String theory in two-dimensions*, hep-th/9108019;
S.R. Das, *The one-dimensional matrix model and string theory*, hep-th/9211085;
A. Jevicki, *Development in 2-D string theory*, hep-th/9309115;
P.H. Ginsparg and G.W. Moore, *Lectures on 2-d gravity and 2-d string theory*, hep-th/9304011;
J. Polchinski, *What is string theory?*, hep-th/9411028.

[2] S.R. Das and A. Jevicki, *String field theory and physical interpretation of $D = 1$ strings*, *Mod. Phys. Lett.* **A 5** (1990) 1639.

[3] J. Polchinski, *Critical behavior of random surfaces in one-dimension*, *Nucl. Phys.* **B 346** (1990) 253.

[4] J. McGreevy and H. Verlinde, *Strings from tachyons: the $c = 1$ matrix reloaded*, *J. High Energy Phys.* **12** (2003) 054 [hep-th/0304224].

[5] I.R. Klebanov, J. Maldacena and N. Seiberg, *D-brane decay in two-dimensional string theory*, *J. High Energy Phys.* **07** (2003) 045 [hep-th/0305159].

[6] J. Polchinski, *Classical limit of $(1 + 1)$-dimensional string theory*, *Nucl. Phys.* **B 362** (1991) 125.

[7] A. Jevicki and B. Sakita, *The quantum collective field method and its application to the planar limit*, *Nucl. Phys.* **B 165** (1980) 511; *Collective field approach to the large-N limit: euclidean field theories*, *Nucl. Phys.* **B 185** (1981) 89;
See B. Sakita, *Quantum theory of many variable systems and fields*, World Scientific, Singapore 1988.

[8] G.'t Hooft, *Dimensional reduction in quantum gravity*, gr-qc/9310026.

[9] L. Susskind, *The world as a hologram*, *J. Math. Phys.* **36** (1995) 6377 [hep-th/9409089].

[10] J.L. Karczmarek and A. Strominger, *Matrix cosmology*, *J. High Energy Phys.* **04** (2004) 055 [hep-th/0309138].

[11] S.R. Das, J.L. Davis, F. Larsen and P. Mukhopadhyay, *Particle production in matrix cosmology*, *Phys. Rev.* **D 70** (2004) 044017 [hep-th/0403275];
P. Mukhopadhyay, *On the problem of particle production in c = 1 matrix model*, *J. High Energy Phys.* **08** (2004) 032 [hep-th/0406029];

[12] A. Sen, *Symmetries, conserved charges and (black) holes in two dimensional string theory*, *J. High Energy Phys.* **12** (2004) 053 [hep-th/0408064].

[13] J. Gomis and A. Kapustin, *Two-dimensional unoriented strings and matrix models*, *J. High Energy Phys.* **06** (2004) 002 [hep-th/0310195].

[14] I. Andrić and L. Jonke, *Duality and quasiparticles in the calogero-sutherland model: some exact results*, hep-th/0010033.

[15] E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, *Planar diagrams*, *Commun. Math. Phys.* **59** (1978) 35.

[16] M.L. Mehta, *Random matrices*, Academic Press, New York 1967.

[17] I. Andrić, A. Jevicki and H. Levine, *On the large-N limit in symplectic matrix models*, *Nucl. Phys.* **B 215** (1983) 307.

[18] A.P. Polychronakos, *Generalized statistics in one dimension*, hep-th/9902157.

[19] I. Andrić and V. Bardek, *1/N corrections in the matrix model and the quantum collective field*, *Phys. Rev.* **D 32** (1985) 1025.

[20] J. Avan and A. Jevicki, *Classical integrability and higher symmetries of collective string field theory*, *Phys. Lett.* **B 266** (1991) 35.

[21] A. Jevicki, *Nonperturbative collective field theory*, *Nucl. Phys.* **B 376** (1992) 75.

[22] I. Andrić and D. Jurman, *Duality and interacting families in models with the inverse-squared interaction*, hep-th/0304006.

[23] I. Andrić and D. Jurman, *Duality in the low dimensional field theory*, *Fortschr. Phys.* **50** (2002) 675.

[24] A.O. Barut and L. Girardello, *New 'coherent' states associated with noncompact groups*, *Commun. Math. Phys.* **21** (1971) 41.

[25] A.P. Polychronakos, *Waves and solitons in the continuum limit of the Calogero- Sutherland model*, *Phys. Rev. Lett.* **74** (1995) 5153 [hep-th/9411054].

[26] I. Andrić, V. Bardek and L. Jonke, *Solitons in the calogero-sutherland collective field model*, *Phys. Lett.* **B 357** (1995) 374 [hep-th/9411136].

[27] J.A. Minahan and A.P. Polychronakos, *Density correlation functions in Calogero-Sutherland models*, *Phys. Rev.* **B 50** (1994) 4236 [hep-th/9404192].

JHEP01(2005)039